§ 3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

We now consider the nonhomogeneous second-order linear DE

$$L[y] = y'' + p(t)\,y' + q(t)\,y = g(t) \quad (*)$$

When $g(t) = 0$, we call the resulting DE the homogeneous DE corresponding to (*).

<u>Thm</u> If Y_1 and Y_2 are solutions of the nonhomogeneous linear DE $L[y] = g(t)$, then $Y_1 - Y_2$ is a solution of the homogeneous DE $L[y] = 0$. If y_1 and y_2 form a fundamental set of solutions to $L[y] = 0$, then

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2 .$$

<u>Proof</u> Y_1 and Y_2 satisfy $L[Y_1]=g(t)$ and $L[Y_2]=g(t)$. Then

$$L[Y_1-Y_2]=L[Y_1]-L[Y_2]$$
$$=g(t)-g(t)$$
$$=0$$

so Y_1-Y_2 is a solution of $L[y]=0$.

Since any solution of $L[y]=0$ is a linear combination of the fundamental solutions

$$Y_1-Y_2=c_1y_1+c_2y_2.$$

<u>Thm</u> The general solution of $L[y]=g(t)$ is of the form

$$y=\varphi(t)=c_1y_1+c_2y_2+Y$$

where y_1 and y_2 form a fundamental set of solutions of $L[y]=0$ and Y is any solution of $L[y]=g(t)$.

Steps for Solving $L[y]=g(t)$

1. Find general solution to homogeneous equation $L[y]=0$

$$y_c = c_1 y_1 + c_2 y_2,$$ called the complementary solution

2. Find solution to $L[y]=g(t)$, y_p called the particular solution.

3. The general is

$$y = y_c + y_p$$

How to find y_p?

Method of Undetermined Coefficients

Idea: • Based on $g(t)$, we make an ansatz on the form of y_p which will involve arbitrary constants

• Find constants so that $L[y_p] = g(t)$.

Ex $y'' - 3y' + 4y = 3e^{2t}$. Find y_p.

We want to find y_p so that

$$y_p'' - 3y_p' + 4y_p = 3e^{2t}$$

So let's try $y_p = Ae^{2t}$. Then

$$4Ae^{2t} - 6Ae^{2t} + 4Ae^{2t} = 3e^{2t}$$

$$2Ae^{2t} = 3e^{2t}$$

$$A = \frac{3}{2}$$

$$y_p = \frac{3}{2}e^{2t}.$$

Remark

1. The same idea holds when $g(t)$ is of a different form.

- Exponential: Guess y_p is proportional to some exponential
- Sine or Cosine: Guess y_p is a linear combination of sine and cosine
- Degree n polynomial: Guess y_p is an n degree polynomial
- Same idea when you have a product/sum of these three types of functions

2. If $g(t) = g_1(t) + g_2(t)$ and $L[Y_1] = g_1(t)$ and $L[Y_2] = g_2(t)$. Then $Y_1 + Y_2$ is

a solution to $L[y] = g(t)$. That is, the determining of y_p can be broken up into several smaller problems when $g(t)$ can be expressed as a sum.

Ex Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos(2t).$$

We can split the problem up into

$$y'' - 3y' - 4y = 3e^{2t} \qquad (1)$$

$$y'' - 3y' - 4y = 2\sin t \qquad (2)$$

$$y'' - 3y' - 4y = -8e^{t}\cos(2t) \qquad (3)$$

First, we guess that $y_p = Ae^{2t}$. Now,

$$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$$

$$-6Ae^{2t} = 3e^{2t}$$

$$A = -\frac{1}{2}.$$

So, $\quad y_{p_1} = \frac{-1}{2} e^{2t}.$

Consider (2). We guess

$$y_{p_2} = A\cos t + B\sin t$$

So

$$y_{p_2}'' - 3y_{p_2}' - 4y_{p_2} = 2\sin t$$

$$-A\cos t - B\sin t - 3(-A\sin t + B\cos t)$$
$$- 4(A\cos t + B\sin t) = 2\sin t$$

$$(-A - 3B - 4A)\cos t + (-B + 3A - 4B)\sin t = 2\sin t$$

$$(-5A - 3B)\cos t + (-5B + 3A)\sin t = 2\sin t$$

$$-5A-3B=0 \qquad -5B+3A=2$$

$$A=\frac{3}{17} \text{ and } B=\frac{-5}{17}$$

so

$$y_{p_2}=\frac{3}{17}\cos t-\frac{5}{17}\sin t \quad .$$

Finally, for (3), we guess

$$y_{p_3}=Ae^t\cos 2t+Be^t\sin 2t$$

Plug y_{p_3} into (3) and we find

$$-10A-2B=-8 \quad \text{and} \quad -10B+2A=0$$

so

$$A=\frac{10}{13} \quad \text{and} \quad B=\frac{2}{13}$$

Therefore, $y_{p_3}=\frac{10}{13}e^t\cos 2t+\frac{2}{13}e^t\sin 2t$

Now, putting everything together,

$$y_p = \frac{-1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^{t}\cos(2t)$$
$$+ \frac{2}{13}e^{t}\sin(2t).$$

Remark Sometimes the guess for the form of the particular solution may overlap with the homogeneous solution. In such cases, multiply the guess by t.

Ex $y'' - 3y' - 4y = 2e^{-t}$

The characteristic equation is

$$r^2 - 3r - 4 = 0$$
$$(r-4)(r+1) = 0$$
$$r = 4 \quad r = -1$$

So
$$y_c = c_1 e^{4t} + c_2 e^{-t}$$
is the solution to the homogeneous problem.

For the nonhomogeneous equation, our initial guess would be Ae^{-t}, but this overlaps with the homogeneous solution so we guess
$$y_p = Ate^{-t}.$$

Now,
$$y_p' = Ae^{-t} - Ate^{-t}$$
and
$$y_p'' = -Ae^{-t} - Ae^{-t} + Ate^{-t}$$
$$= -2Ae^{-t} + Ate^{-t}$$

Now,

$$y_p'' - 3y_p' - 4y_p = 2e^{-t}$$

$$-2Ae^{-t} + Ate^{-t} - 3Ae^{-t} + 3Ate^{-t} - 4Ate^{-t} = 2e^{-t}$$

$$-5Ae^{-t} = 2e^{-t}$$

$$A = \frac{-2}{5}$$

So $y_p = \frac{-2}{5} t e^{-t}$.

§ 3.6 Variation of Parameters

Consider the second-order DE

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

and assume we know the general solution of the homogeneous equation to be

$$y_c = c_1 y_1 + c_2 y_2 .$$

Now, we suppose

$$y_p = u_1(t)\,y_1 + u_2(t)\,y_2$$

where u_1 and u_2 are to-be-determined.

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

Note that we have 1 equation for two unknown functions, so there may be many choices of u_1 and u_2. We add the condition that

$$u_1' y_1 + u_2' y_2 = 0$$

so

$$y_p' = u_1 y_1' + u_2 y_2'$$

Next

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

Plugging in to the DE,

$$u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' + p(u_1 y_1' + u_2 y_2')$$
$$+ q(u_1 y_1 + u_2 y_2) = g$$

$$u_1(y_1'' + p y_1' + q y_1) + u_2(y_2'' + p y_2' + q y_2)$$
$$+ u_1' y_1' + u_2' y_2' = g$$

so

$$u_1' y_1' + u_2' y_2' = g$$

Since y_1 and y_2 are solutions to the homogeneous equation. Therefore, we have the system of equations

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g(t)$$

and in matrix form

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

$$= \frac{1}{y_1 y_2' - y_2 y_1'} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

$$= \frac{1}{W[y_1, y_2]} \begin{bmatrix} -y_2 g(t) \\ y_1 g(t) \end{bmatrix}$$

and so

$$u_1' = \frac{1}{W[y_1, y_2]} (-y_2 g(t))$$

$$u_2' = \frac{1}{W[y_1, y_2]} y_1 g(t)$$

Thus,

$$u_1 = \int \frac{-y_2 g(t)}{W[y_1, y_2]} \, dt + c_1$$

$$u_2 = \int \frac{y_1 g(t)}{W[y_1, y_2]} \, dt + c_2$$

and our general solution is

$$y = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2 .$$

Ex Find the general solution of

$$y'' + 4y = 8\tan t \quad , \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

First, we find the solution to the homogeneous eqn.

$$r^2 + 4 = 0$$

$$r = \pm 2i$$

$$y_c = C_1 \cos(2t) + C_2 \sin(2t)$$

Now, we suppose

$$y_p = u_1 \cos(2t) + u_2 \sin(2t)$$

and so

$$y_p' = -2u_1\sin(2t) + 2u_2\cos(2t)$$
$$+ u_1'\cos(2t) + u_2'\sin(2t)$$

We assume

$$u_1'\cos(2t) + u_2'\sin(2t) = 0$$

So

$$y_p' = -2u_1\sin(2t) + 2u_2\cos(2t).$$

Now

$$y_p'' = -2u_1'\sin(2t) - 4u_1\cos(2t)$$
$$+ 2u_2'\cos(2t) - 4u_2\sin(2t)$$

Plugging y_p into the nonhomogeneous equation, we have

$$y_p'' + 4y = 8\tan t$$

$$-2u_1'\sin(2t) - 4u_1\cos(2t) + 2u_2'\cos(2t)$$
$$- 4u_2\sin(2t) + 4u_1\cos(2t) + 4u_2\sin(2t) = 8\tan t$$

$$-2u_1' \sin(2t) + 2u_2' \cos(2t) = 8\tan t$$

and so we have the system

$$u_1' \cos(2t) + u_2' \sin(2t) = 0$$
$$-2u_1' \sin(2t) + 2u_2' \cos(2t) = 8\tan t$$

$$\begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 8\tan t \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{2\cos^2(2t) + 2\sin^2(2t)} \begin{bmatrix} 2\cos(2t) & -\sin(2t) \\ 2\sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} 0 \\ 8\tan t \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -8\sin(2t)\tan t \\ 8\cos(2t)\tan t \end{bmatrix}$$

$$u_1' = -4\sin(2t)\tan t \qquad u_2' = 4\cos(2t)\tan t$$